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Constructing a continuously differentiable exact augmented Lagrangian function for nonlinear semidefinite programming

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Abstract

In this note, we will construct a continuouly differentiable exact augmented Lagrangian function for nonlinear semidefinite programming problems. This function is defined on the product space of the problem's variables and of the multipliers. The unconstrained minimization of the proposed exact augmented Lagrangian gives a solution of the original problem, when the penalty parameter is large enough. We will show that the exactness property holds when the nondegeneracy condition is assumed.

Keywords: Nonlinear semidefinite programming, exact augmented Lagrangian functions, exact penalty functions, nondegeneracy.

1 Introduction

The following *nonlinear semidefinite programming* (NSDP) problem is considered:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & G(x) \in \mathbb{S}^m_+, \end{array} \qquad \text{(NSDP)}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ and $G\colon \mathbb{R}^n \to \mathbb{S}^m$ are twice continuously differentiable functions, $\mathbb{S}^m$ is the linear space of all real symmetric matrices of dimension $m \times m$, and $\mathbb{S}^m_+$ is the cone of all positive semidefinite matrices in $\mathbb{S}^m$. Here, we omit equality constraints just for simplicity. The above formulation is considerably new, but it was already used in many application fields, like control theory [2, 12], structural optimization [16, 18], truss design problems [3], and finance [17]. The research associated to NSDP is, however, relatively scarce, if we compare to the particular case of linear semidefinite programming problems.

We refer to [22] for a survey of numerical methods for NSDP problems. In particular, it describes the augmented Lagrangian, the sequential quadratic programming, and the primal-dual interior point methods. In this paper, we propose another method for solving

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general NSDP problems. More precisely, we construct a continuously differentiable *exact augmented Lagrangian* function for NSDP. By exact, we mean that an unconstrained minimization of this function recovers a solution of the original problem, when a certain penalty parameter is large enough. The exact augmented Lagrangian function also differs from the exact penalty one. In fact, the former is defined on the product space of the problem's variables and of the Lagrange multipliers, while the latter is defined in the same space of the original constrained problem. Such a terminology is actually used in the literature of the traditional *nonlinear programming* (NLP) [7, 8].

Exact augmented Lagrangian functions were introduced by Di Pillo and Grippo in [7] and [8], respectively for equality-constrained and for inequality-constrained NLP problems. Further investigations had been done in [4, 9, 10, 11, 19]. Recalling that NSDP problems extend NLP problems, here we give the first step towards the augmented Lagrangian method for NSDP. The proposed function is basically the classical augmented Lagrangian function for NSDP problems, given in [6, 21], with an additional term that guarantees the exactness property. Such term requires a function that estimates the Lagrange mutipliers associated to a point. This estimator was originally given in [14], and extended further to NSDP in [15]. We will show that the proposed function is in fact exact, if the nondegeneracy condition is satisfied in the feasible set of the NSDP.

We finally observe that a more general class of these exact augmented Lagrangian functions for NSDP is given in [13]. In fact, our paper is just an easy note associated to it, so all the proofs of the results described here can be seen in this original manuscript. This paper is organized as follows. In Section 2, we recall basic notations and definitions. The exact augmented Lagrangian function is constructed in Section 3, and the exactness results are given in Section 4. We conclude in Section 5, with some final remarks and future works.

2 Preliminaries

Let us first present the main notations. We use x_i and Z_{ij} to denote the ith element of a vector $x \in \mathbb{R}^r$ and (i,j) entry (ith row and jth column) of a matrix $Z \in \mathbb{S}^s$, respectively. We also use the notation $[x_i]_{i=1}^r$ and $[Z_{ij}]_{i,j=1}^s$ to denote x and Z, respectively. For a function $p\colon \mathbb{R}^s \to \mathbb{R}$, its gradient and Hessian at a point $x \in \mathbb{R}^s$ are given by $\nabla p(x) \in \mathbb{R}^s$ and $\nabla^2 p(x) \in \mathbb{R}^{s\times s}$, respectively. For $q\colon \mathbb{S}^\ell \to \mathbb{R}$, $\nabla q(Z)$ denotes the matrix with (i,j) term given by the partial derivatives $\partial q(Z)/\partial Z_{ij}$. If $\psi\colon \mathbb{R}^s \times \mathbb{S}^\ell \to \mathbb{R}$, then its gradient at $(x,Z) \in \mathbb{R}^s \times \mathbb{S}^\ell$ with respect to x and Y are denoted by $\nabla_x \psi(x,Y) \in \mathbb{R}^s$ and $\nabla_Y \psi(x,Y) \in \mathbb{S}^\ell$, respectively. Similary, the Hessian of ψ at (x,Z) with respect to x is written as $\nabla^2_{xx}\psi(x,Y)$.

For any linear operator $\mathcal{G}\colon \mathbb{R}^s \to \mathbb{S}^\ell$ defined by $\mathcal{G}v = \sum_{i=1}^s v_i \mathcal{G}_i$ with $\mathcal{G}_i \in \mathbb{S}^\ell$, $i = 1,\dots,s$, and $v \in \mathbb{R}^s$, the adjoint operator $\mathcal{G}^*$ is defined by

$$\mathcal{G}^* Z = (\langle \mathcal{G}_1, Z\rangle, \dots, \langle \mathcal{G}_s, Z\rangle)^\top, \quad Z \in \mathbb{S}^\ell.$$

Moreover, given a mapping $\mathcal{G}\colon \mathbb{R}^s \to \mathbb{S}^\ell$, its derivative at a point $x \in \mathbb{R}^s$ is denoted by

$\nabla \mathcal{G}(x)\colon \mathbb{R}^s \to \mathbb{S}^\ell$ and defined by

$$\nabla \mathcal{G}(x)v = \sum_{i=1}^{s} v_i \frac{\partial \mathcal{G}(x)}{\partial x_i}, \quad v \in \mathbb{R}^s,$$

where $\partial \mathcal{G}(x)/\partial x_i \in \mathbb{S}^\ell$ are the partial derivative matrices.

One important operator that is necessary when dealing with NSDP problems is the *Jordan product* associated to the space $\mathbb{S}^m$. For any $Y, Z \in \mathbb{S}^m$, it is defined by

$$Y \circ Z := \frac{YZ + ZY}{2}.$$

Taking $Y \in \mathbb{S}^m$, we also denote by $\mathcal{L}_Y\colon \mathbb{S}^m \to \mathbb{S}^m$ the linear operator given by

$$\mathcal{L}_Y(Z) := Y \circ Z.$$

Since we are only considering the space $\mathbb{S}^m$ of symmetric matrices, we have $\mathcal{L}_Y(Z) = \mathcal{L}_Z(Y)$.

Now, let the trace of $Z \in \mathbb{S}^m$ be given by $\operatorname{tr}(Z) := \sum_{i=1}^{s} Z_{ii}$ and define $\langle Y, Z\rangle := \operatorname{tr}(YZ)$ as the inner product of symmetric matrices Y and Z. Then, define $L\colon \mathbb{R}^n \times \mathbb{S}^m \to \mathbb{R}$ as the Lagrangian function associated to problem (NSDP), that is,

$$L(x, \Lambda) := f(x) - \langle G(x), \Lambda\rangle.$$

The pair $(x, \Lambda) \in \mathbb{R}^n \times \mathbb{S}^m$ satisfies the *Karush-Kuhn-Tucker (KKT) conditions* of problem (NSDP) (or, it is a KKT pair) if the following conditions hold:

$$\begin{array}{rcl} \nabla_x L(x, \Lambda) & = & 0, \\ \Lambda \circ G(x) & = & 0, \\ G(x) & \in & \mathbb{S}^m_+, \\ \Lambda & \in & \mathbb{S}^m_+, \end{array}$$

where $\nabla_x L(x, \Lambda)$ denotes the gradient of L with respect to x, that is,

$$\nabla_x L(x, \Lambda) = \nabla f(x) - \nabla G(x)^* \Lambda.$$

The above conditions are necessary for optimality under a constraint qualification. Moreover, it can be shown that the condition $\Lambda \circ G(x) = 0$ can be replaced by $\langle \Lambda, G(x)\rangle = 0$ or $\Lambda G(x) = 0$ because $G(x) \in \mathbb{S}^m_+$ and $\Lambda \in \mathbb{S}^m_+$ hold [22, Section 2].

In this paper, we will replace the problem (NSDP) with the following problem, which is just a nonlinear programming:

$$\begin{array}{ll} \underset{x,\Lambda}{\text{minimize}} & \Psi_c(x, \Lambda) \\ \text{subject to} & (x, \Lambda) \in \mathbb{R}^n \times \mathbb{S}^m, \end{array} \tag{1}$$

where $\Psi_c\colon \mathbb{R}^n \times \mathbb{S}^m \to \mathbb{R}$, and $c > 0$ is a penalty parameter. Observe that the above problem is unconstrained, with both x and Λ as variables. As usual, we say that a point $x \in \mathbb{R}^n$ is *stationary* of Ψ_c when $\nabla \Psi_c(x) = 0$. We use $G_{\text{NLP}}(c)$ and $L_{\text{NLP}}(c)$ to denote the sets of global and local minimizers, respectively, of the above problem. We also define G_{NSDP} and L_{NSDP} as the set of global and local minimizers of problem (NSDP), respectively. Using such notations, we present below the formal definition of exact augmented Lagrangian functions.

Definition 1. *A function* $\Psi_c\colon \mathbb{R}^n \times \mathbb{S}^m \to \mathbb{R}$ *is called an* exact augmented Lagrangian *function associated to* (NSDP) *if, and only if, there exists* $\hat{c} > 0$ *satisfying the following:*

(a) For all $c > \hat{c}$, *if* $(\bar{x}, \bar{\Lambda}) \in G_{NLP}(c)$, *then* $\bar{x} \in G_{NSDP}$ *and* $\bar{\Lambda}$ *is a corresponding Lagrange multiplier. Conversely, if* $\bar{x} \in G_{NSDP}$ *with* $\bar{\Lambda}$ *as a corresponding Lagrange multiplier, then* $(\bar{x}, \bar{\Lambda}) \in G_{NLP}(c)$ *for all* $c > \hat{c}$.

(b) For all $c > \hat{c}$, *if* $(\bar{x}, \bar{\Lambda}) \in L_{NLP}(c)$, *then* $\bar{x} \in L_{NSDP}$ *and* $\bar{\Lambda}$ *is a corresponding Lagrange multiplier.*

Basically, the above definition shows that Ψ_c is an exact augmented Lagrangian function when there are equivalence between the global minimizers, and if all local solutions of (1) are local solutions of (NSDP), for penalty parameters greater than a threshold value. It means that the original constrained conic problem (NSDP) can be replaced with an uncontrained nonlinear programming problem (1) when the penalty parameter is chosen appropriately. In order to construct such an exact augmented Lagrangian function, we will suppose that the following assumption holds in the whole paper. It is well-known that the nondegeneracy condition, defined below, extends the classical linear independence constraint qualification for nonlinear programming [5, 20].

Assumption 2. *Every* $x \in \mathbb{R}^n$ *feasible for* (NSDP) *is* nondegenerate, *that is,*

$$\mathbb{S}^m = \mathrm{lin}\mathcal{T}_{\mathbb{S}^m_+}(G(x)) + \mathrm{Im}\,\nabla G(x),$$

where $\mathcal{T}_{\mathbb{S}^m_+}(G(x))$ *denotes the tangent cone of* $\mathbb{S}^m_+$ *at* $G(x)$, $\mathrm{Im}\,\nabla G(x)$ *is the image of the linear map* $\nabla G(x)$, *and* lin *means lineality space.*

3 The proposed augmented Lagrangian function

The exact augmented Lagrangian function considered in [8] takes into account an estimation of the Lagrange multipliers, given in [14]. An extension of it for NSDP problems were proposed in [15]. Given $x \in \mathbb{R}^n$, we solve the following unconstrained problem:

$$\begin{array}{ll} \underset{\Lambda}{\text{minimize}} & \|\nabla_x L(x,\Lambda)\|^2 + \zeta_1^2\|\mathcal{L}_{G(x)}(\Lambda)\|_F^2 + \zeta_2^2 r(x)\|\Lambda\|_F^2 \\ \text{subject to} & \Lambda \in \mathbb{S}^m, \end{array} \tag{2}$$

where $\zeta_1, \zeta_2 \in \mathbb{R}$ are positive scalars, $\|\cdot\|$ denotes the Euclidean norm, $\|\cdot\|_F$ is the Frobenius norm, and $r\colon \mathbb{R}^n \to \mathbb{R}$ denotes the residual function associated to the feasible set, that is,

$$r(x) := \frac{1}{2}\|P_{\mathbb{S}^m_+}(-G(x))\|_F^2 = \frac{1}{2}\|P_{\mathbb{S}^m_+}(G(x)) - G(x)\|_F^2,$$

with $P_{\mathbb{S}^m_+}$ denoting the projection onto $\mathbb{S}^m_+$. Observe that $r(x) = 0$ if, and only if, x is feasible for (NSDP).

Lemma 3. *[15, Lemma 2.2 and Proposition 2.3] Suppose that Assumption 2 holds. For any* $x \in \mathbb{R}^n$, *define* $N\colon \mathbb{R}^n \to \mathbb{S}^m$ *as*

$$N(x) := \nabla G(x)\nabla G(x)^* + \zeta_1^2\mathcal{L}_{G(x)}^2 + \zeta_2^2\, r(x) I, \tag{3}$$

where I *denotes the identity matrix. Then, the following statements are true.*

(a) *$N(\cdot)$ is continuosly differentiable and for all $x \in \mathbb{R}^n$, the matrix $N(x)$ is positive definite.*

(b) *The solution of problem* (2) *is unique and it is given by*

$$\Lambda(x) = N(x)^{-1}\nabla G(x)\nabla f(x). \tag{4}$$

(c) *If $(x, \Lambda) \in \mathbb{R}^n \times \mathbb{S}^m$ is a KKT pair of* (NSDP)*, then $\Lambda(x) = \Lambda$.*

(d) *The operator $\Lambda(\cdot)$ is continuously differentiable, and $\nabla\Lambda(x) = N(x)^{-1}Q(x)$, where*

$$\begin{aligned} Q(x) \;&:=\; \nabla^2 G(x)\nabla_x L(x, \Lambda(x)) + \nabla G(x)\nabla^2_{xx} L(x, \Lambda(x)) \\ &\quad - \zeta_1^2 \nabla_x \left[\mathcal{L}^2_{G(x)}(\Lambda(x))\right] - \zeta_2^2 \nabla r(x)\Lambda(x). \end{aligned}$$

Based on the above lemma, we propose the following augmented Lagrangian function.

$$L_c(x, \Lambda) := f(x) + \frac{1}{2c}\left(\|P_{\mathbb{S}^m_+}(\Lambda - cG(x))\|_F^2 - \|\Lambda\|_F^2\right) + \|N(x)(\Lambda(x) - \Lambda)\|_F^2. \tag{5}$$

Note that it is equivalent to the augmented Lagragian function for nonlinear semidefinite programs, except for the last term $\|N(x)(\Lambda(x) - \Lambda)\|_F^2$. From (3) and (4), observe that

$$\begin{aligned} N(x)(\Lambda(x) - \Lambda) \;&=\; \nabla G(x)\nabla f(x) - \nabla G(x)\nabla G(x)^*\Lambda - \zeta_1^2\mathcal{L}^2_{G(x)}(\Lambda) - \zeta_2^2 r(x)\Lambda \\ &=\; \nabla G(x)\nabla_x L(x, \Lambda) - \zeta_1^2\mathcal{L}^2_{G(x)}(\Lambda) - \zeta_2^2 r(x)\Lambda. \end{aligned} \tag{6}$$

Also, consider the following auxiliar function $Y_c\colon \mathbb{R}^n \times \mathbb{S}^m \to \mathbb{S}^m$ defined by

$$Y_c(x, \Lambda) := P_{\mathbb{S}^m_+}\left(\frac{\Lambda}{c} - G(x)\right) - \frac{\Lambda}{c}.$$

Then, the gradient of $L_c(x, \Lambda)$ with respect to x is given by

$$\begin{aligned} \nabla_x L_c(x, \Lambda) \;&=\; \nabla f(x) - \nabla G(x)^* P_{\mathbb{S}^m_+}(\Lambda - cG(x)) + 2K(x, \Lambda)^* N(x)(\Lambda(x) - \Lambda) \\ &=\; \nabla_x L(x, \Lambda) - c\nabla G(x)^* Y_c(x, \Lambda) + 2K(x, \Lambda)^* N(x)(\Lambda(x) - \Lambda), \end{aligned}$$

with

$$\begin{aligned} K(x, \Lambda) \;&:=\; \nabla_x\left[N(x)(\Lambda(x) - \Lambda)\right] \\ &=\; \nabla_x\left[\nabla G(x)\nabla_x L(x, \Lambda) - \zeta_1^2\mathcal{L}^2_{G(x)}(\Lambda) - \zeta_2^2 r(x)\Lambda\right], \end{aligned}$$

where the second equality follows from (6). Some calculations show that

$$\begin{aligned} \nabla_x L_c(x, \Lambda) \;&=\; \nabla_x L(x, \Lambda) - c\nabla G(x)^* Y_c(x, \Lambda) + 2\nabla^2_{xx} L(x, \Lambda)\nabla G(x)^* N(x)(\Lambda(x) - \Lambda) \\ &\quad + 2\left[\left\langle \frac{\partial^2 G(x)}{\partial x_i x_j}, N(x)(\Lambda(x) - \Lambda)\right\rangle\right]_{i,j=1}^n \nabla_x L(x, \Lambda) \\ &\quad - 2\zeta_1^2\left[\left\langle \frac{\partial G(x)}{\partial x_i} \circ (G(x) \circ \Lambda) + G(x) \circ \left(\frac{\partial G(x)}{\partial x_i} \circ \Lambda\right), N(x)(\Lambda(x) - \Lambda)\right\rangle\right]_{i=1}^n \\ &\quad - 2\zeta_2^2\langle \Lambda, N(x)(\Lambda(x) - \Lambda)\rangle\nabla r(x). \end{aligned}$$

Moreover, the gradient of $L_c(x, \Lambda)$ with respect to Λ can be written as follows:

$$\nabla_\Lambda L_c(x, \Lambda) = Y_c(x, \Lambda) - 2N(x)^2(\Lambda(x) - \Lambda).$$

4 Exactness results

In this section, we will first establish the relation between the KKT points of the original (NSDP) problem with the following unconstrained one:

$$\begin{array}{ll} \underset{x,\Lambda}{\text{minimize}} & L_c(x,\Lambda) \\ \text{subject to} & (x,\Lambda) \in \mathbb{R}^n \times \mathbb{S}^m. \end{array} \tag{7}$$

We refer to [13] for details and proofs of the subsequent results. As it can be seen in the next three propositions, a KKT pair of (NSDP) is stationary of (7), but the converse implication only holds when the penalty parameter is greater than a threshold value. Moreover, as in the classical augmented Lagrangian methods or exact penalty methods [1], we may also end up with a stationary point of the residual function r that is infeasible for (NSDP).

Proposition 4. *If $(x,\Lambda) \in \mathbb{R}^n \times \mathbb{S}^m$ is a KKT pair of* (NSDP), *then, for all $c > 0$, it is stationary of* (7), *that is, $\nabla_x L_c(x,\Lambda) = 0$ and $\nabla_\Lambda L_c(x,\Lambda) = 0$.*

Proposition 5. *Let $\hat{x} \in \mathbb{R}^n$ be feasible for* (NSDP). *Assume that there exist $\{x^k\} \subset \mathbb{R}^n$, $\{\Lambda_k\} \subset \mathbb{S}^m$, and $\{c_k\} \subset \mathbb{R}_{++}$ with $x_k \to \hat{x}$ and $c_k \to \infty$ such that (x_k,Λ_k) is stationary of* (7) *for all k. Then, there is $\hat{k} > 0$ such that (x^k,Λ_k) is KKT of* (NSDP) *for all $k > \hat{k}$.*

Proposition 6. *Let $\{x^k\} \subset \mathbb{R}^n$, $\{\Lambda_k\} \subset \mathbb{S}^m$, and $\{c_k\} \subset \mathbb{R}_{++}$ be sequences such that $c_k \to \infty$ and (x_k,Λ_k) is stationary of* (7) *for all k. Assume that there is a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ such that $x^{k_j} \to \hat{x}$ for some $\hat{x} \in \mathbb{R}^n$. Then, either there exists $\hat{k} > 0$ such that (x^{k_j},Λ_{k_j}) is a KKT pair of* (NSDP) *for all $k_j > \hat{k}$, or $\hat{x}$ is a stationary point of the residual function r that is infeasible for* (NSDP).

We now use the notations G_{NSDP} (L_{NSDP}) and $G_{\text{NLP}}(c)$ ($L_{\text{NLP}}(c)$) for the sets of global (local) minimizers of problems (NSDP) and (7), respectively. The following theorems show that the proposed function L_c given in (5) is in fact an exact augmented Lagrangian function.

Theorem 7. *Let $\{x^k\} \subset \mathbb{R}^n$, $\{\Lambda_k\} \subset \mathbb{S}^m$, and $\{c_k\} \subset \mathbb{R}_{++}$ be sequences such that $c_k \to \infty$ and $(x_k,\Lambda_k) \in L_{NLP}(c_k)$ for all k. Assume that there is a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ such that $x^{k_j} \to \hat{x}$ for some $\hat{x} \in \mathbb{R}^n$. Then, either there exists $\hat{k} > 0$ such that $x^{k_j} \in L_{NSDP}$, with an associated Lagrange multiplier Λ_{k_j} for all $k_j > \hat{k}$, or $\hat{x}$ is a stationary point of the residual function r that is infeasible for* (NSDP).

Theorem 8. *Assume that there exists $\bar{c} > 0$ such that $\cup_{c \geq \bar{c}} G_{NLP}(c)$ is bounded. Then, there exists $\hat{c} > 0$ such that $G_{NLP}(c) = G_{NSDP}$ for all $c \geq \hat{c}$.*

5 Final remarks

We proposed an exact augmented Lagrangian function for general nonlinear semidefinite programming problems, and we proved the exactness result under the nondegeneracy condition. Thus, by solving the unconstrained problem (7) with an appropriate penalty parameter, it is possible to recover solutions of (NSDP). The nonlinear programming problem (7) can be solved using a second-order method, like the semismooth Newton. In such a case, convergence results should be established, and a way to avoid the third-order terms of the problem data, that appear in the Hessian of L_c, should be also considered. These facts and the numerical experiments are under investigation.

References

[1] R. Andreani, E. H. Fukuda, and P. J. S. Silva. A Gauss-Newton approach for solving constrained optimization problems using differentiable exact penalties. *Journal of Optimization Theory and Applications*, 156(2):417–449, 2013.

[2] P. Apkarian, D. Noll, and H. D. Tuan. Fixed-order H_∞ control design via a partially augmented Lagrangian method. *International Journal of Robust and Nonlinear Control*, 13(12):1137–1148, 2003.

[3] A. Ben-Tal, F. Jarre, M. Kočvara, A. Nemirovski, and J. Zowe. Optimal design of trusses under a nonconvex global buckling constraint. *Optimization and Engineering*, 1(2):189–213, 2000.

[4] D. P. Bertsekas. *Constrained Optimization and Lagrange Multipliers Methods.* Academic Press, New York, 1982.

[5] J. F. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Problems.* Springer-Verlag, New York, 2000.

[6] R. Correa and H. Ramírez C. A global algorithm for nonlinear semidefinite programming. *SIAM Journal on Optimization*, 15(1):303–318, 2004.

[7] G. Di Pillo and L. Grippo. A new class of augmented Lagrangians in nonlinear programming. *SIAM Journal on Control and Optimization*, 17(5):618–628, 1979.

[8] G. Di Pillo and L. Grippo. A new augmented Lagrangian function for inequality constraints in nonlinear programming. *Journal of Optimization Theory and Applications*, 36(4):495–519, 1982.

[9] G. Di Pillo, G. Liuzzi, S. Lucidi, and L. Palagi. An exact augmented Lagrangian function for nonlinear programming with two-sided constraints. *Computational Optimization and Applications*, 25:57–83, 2003.

[10] G. Di Pillo and S. Lucidi. On exact augmented Lagrangian functions in nonlinear programming. In G. Di Pillo and F. Giannessi, editors, *Nonlinear Optimization and Applications*, pages 85–100. Springer, Boston, MA, 1996.

[11] G. Di Pillo and S. Lucidi. An augmented Lagrangian function with improved exactness properties. *SIAM Journal on Optimization*, 12(2):376–406, 2001.

[12] B. Fares, P. Apkarian, and D. Noll. An augmented Lagrangian method for a class of LMI-constrained problems in robust control theory. *International Journal of Control*, 74(4):348–360, 2001.

[13] E. H. Fukuda and B. F. Lourenço. Exact augmented Lagrangian functions for nonlinear semidefinite programming. 2016. Submitted.

[14] T. Glad and E. Polak. A multiplier method with automatic limitation of penalty growth. *Mathematical Programming*, 17(2):140–155, 1979.

[15] L. Han. The differentiable exact penalty function for nonlinear semidefinite programming. *Pacific Journal of Optimization*, 10(2):285–303, 2014.

[16] Y. Kanno and I. Takewaki. Sequential semidefinite program for maximum robustness design of structures under load uncertainty. *Journal of Optimization Theory and Applications*, 130(2):265–287, 2006.

[17] H. Konno, N. Kawadai, and D. Wu. Estimation of failure probability using semi-definite logit model. *Computational Management Science*, 1(1):59–73, 2003.

[18] M. Kočvara and M. Stingl. Solving nonconvex SDP problems of structural optimization with stability control. *Optimization Methods & Software*, 19(5):595–609, 2004.

[19] S. Lucidi. New results on a class of exact augmented Lagrangians. *Journal of Optimization Theory and Applications*, 58(2), 1988.

[20] A. Shapiro. First and second order analysis of nonlinear semidefinite programs. *Mathematical Programming*, 77(1):301–320, 1997.

[21] D. Sun, J. Sun, and L.W. Zhang. The rate of convergence of the augmented Lagrangian method for nonlinear semidefinite programming. *Mathematical Programming*, 114(2):349–391, 2008.

[22] H. Yamashita and H. Yabe. A survey of numerical methods for nonlinear semidefinite programming. *Journal of the Operations Research Society of Japan*, 58(1):24–60, 2015.